Achieving Envy-Freeness through Items Sale

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Abstract

We consider a fair division setting of allocating indivisible items to a set of agents. In order to cope with the well-known impossibility results related to the non-existence of envy-free allocations, we allow the option of selling some of the items so as to compensate envious agents with monetary rewards. In fact, this approach is not new in practice, as it is applied in some countries in inheritance or divorce cases. A drawback of this approach is that it may create a value loss, since the market value derived by selling an item can be less than the value perceived by the agents. Therefore, given the market values of all items, a natural goal is to identify which items to sell so as to arrive at an envy-free allocation, while at the same time maximizing the overall social welfare. Our work is focused on the algorithmic study of this problem, and we provide both positive and negative results on its approximability. When the agents have a commonly accepted value for each item, our results show a sharp separation between the cases of two or more agents. In particular, we establish a PTAS for two agents, and we complement this with a hardness result, that for three or more agents, the best approximation guarantee is provided by essentially selling all items. This hardness barrier, however, is relieved when the number of distinct item values is constant, as we provide an efficient algorithm for any number of agents. We also explore the generalization to heterogeneous valuations, where the hardness result continues to hold, and where we provide positive results for certain special cases.

Keywords

Fair Item Allocation, Approximation Algorithms, Envy-freeness, Markets.

1. Introduction

Fair division refers to the algorithmic question of allocating resources or tasks to a set of agents according to some justice criteria. It is by now a prominent area within Algorithmic Game Theory and Computational Social Choice, [1, Part II], dating back to the origins of the civil society. One of the most natural and well studied notions of fairness is *envy-freeness* [2]: a division is envy-free if everyone thinks that her share is at least as valuable as the share of any other agent. In the presence of indivisible items however, obtaining an envy-free allocation is much more challenging [3], and it is well known that, in the majority of cases, envy-free divisions do not exist.

An approach that has been followed by several works, in order to cope with these existential issues, is to focus on relaxations of envy-freeness (for more on this we refer to our related

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work section). Another natural direction that comes into mind is to insist on envy-freeness but provide some compensation (e.g., monetary) to the agents who may feel unhappy by a proposed division. Such models have been considered in the literature, where money is either coming as an external subsidy from a third party or is already part of the initial endowment. Under this setting, [4] investigated the question of determining the minimum amount of money needed to obtain an envy-free division.

In this work, we also allow for monetary rewards, but we choose a different approach, as already initiated in [5]: we require that the money used to compensate the envious agents has to be raised from the set of available items, by selling some of them. This is what happens, for instance, in inheritance division. To provide some examples, as stated in Article n.9 of the New York Laws - Real Property Actions and Article n.720 of the Italian Civil Code, whenever an agreement is not possible, part of the inheritance can be sold through an auction. The same practice is also used in divorce settlements. Clearly, envy-freeness is then always feasible by selling, if needed, the whole inheritance, and equally sharing the proceeds. However, the amount of money raised by this process can be fairly below the real value of the sold items for at least two reasons. First, the bidders who participate in this type of auctions usually aim at winning items at very low prices; secondly, running an auction bears organizational costs which need to be subtracted from the proceeds. Thus, it is in the interest of the heirs to determine an envy-free division by selling assets with as little value loss as possible. This gives rise to an interesting optimization problem of determining which items to sell so as to arrive at an envy-free allocation, with optimal social welfare. Algorithmically, this question has been largely unexplored, with the exception of a particular case handled in [5].

2. Model and Definitions

We consider a set $[m] := \{1, \ldots, m\}$ of m indivisible items to be allocated to a set $[n]$ of n agents. We assume that for every item j, there is a commonly accepted value $v(j)$, by all agents[1]. The vector $\boldsymbol{v} = (v(1), \ldots, v(m))$ induces an additive valuation function $v : 2^{[m]} \mapsto \mathbb{N}$, so that for every subset $S \subseteq [m]$, the value of S is $v(S) = \sum_{j \in S} v(j)$.

An additional choice, instead of allocating all items to the agents, is to sell some of them in exchange of money. The rationale here is that if allocating all items cannot result in a fair allocation, we could use monetary compensations from sold items to achieve a more acceptable outcome. This may come at some value loss, since selling an item in the market can lead to a lower price than the value perceived by the agents. In particular, we assume that we are given a *market value* vector $\boldsymbol{v}_0 = (v_0(1), \ldots, v_0(m))$, so that $v_0(j)$ is the monetary amount that can be obtained by selling item j, with $v_0(j) \leq v(j)$, for every $j \in [m]$. Viewing the vector $\boldsymbol{v}_0$ as inducing an (alternative) additive valuation function, we have that for every $S \subseteq [m]$, the money obtained by selling the items of S is equal to $v_0(S) = \sum_{j \in S} v_0(j)$.

Given an instance defined by a tuple $(n, m, \boldsymbol{v}, \boldsymbol{v}_0)$, an *allocation with items sale* is a partition of $[m]$ into $n + 1$ subsets $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$, such that, for each $i \in [n]$, X_i is the

[1]We start with this modeling choice, as it is common in inheritance or divorce settlements, that items such as land properties or cars have a common value to the agents. In section “Contribution”, we also explore extensions beyond this assumption.

bundle allocated to agent i and X_0 is the set of items which are sold. Hence, the money made from $\boldsymbol{X}$ is $v_0(X_0)$. The *social welfare* of an allocation with items sale is given by $SW(\boldsymbol{X}) = v_0(X_0) + \sum_{i\in[n]} v(X_i)$. We will say that an allocation $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ is an *envy-free allocation with items sale* (from now on, simply EF-IS), if there exists a split of the money $v_0(X_0)$ into n amounts $\mu_1, \ldots, \mu_n$, such that, for any two agents $i, i' \in [n]$, $v(X_i) + \mu_i \geq v(X_{i'}) + \mu_{i'}$. Since this needs to hold for any pair of agents, we can simplify the definition of EF-IS as follows. Define the maximum envy of an agent i under an allocation $\boldsymbol{X}$ as $e_i^{max}(\boldsymbol{X}) = \max_{i'\in[n]}\{v(X_{i'})\} - v(X_i)$ (note that, as i' can be also equal to i, $e_i^{max}(\boldsymbol{X}) \geq 0$). Then, an allocation is EF-IS if and only if $v_0(X_0) \geq \sum_{i\in[n]} e_i^{max}(\boldsymbol{X})$. Hence, whenever the above equation holds, it means that there is enough money to compensate all agents having non-zero maximum envy.

We observe that an EF-IS allocation always exists: simply sell all items and share equally all the money. We call this allocation, the *basic* EF-IS allocation. Therefore, this gives rise to the natural optimization problem of finding the best EF-IS allocation in terms of social welfare. This constitutes the focus of our work, and we define it formally below.

BEST-EF-IS problem: Given an instance $(n, m, \boldsymbol{v}, \boldsymbol{v}_0)$ on n agents, m items, value vector $\boldsymbol{v}$ and market value vector $\boldsymbol{v}_0$, find an allocation $\boldsymbol{X} = (X_0, X_1, \ldots, X_n)$ that is EF-IS and attains maximum social welfare.

3. Contribution

In this work, we embark on a thorough investigation of algorithmic and complexity questions for our problem and provide an almost tight set of results.

We first define a parameter that plays a fundamental role in the majority of our results. Given an instance $(n, m, \boldsymbol{v}, \boldsymbol{v}_0)$, let $\alpha := \min_{j\in[m]:v(j)>0}\left\{\frac{v_0(j)}{v(j)}\right\} \in [0, 1]$, be the largest discrepancy between the market value and the commonly accepted value of any item. We show the NP-hardness of our problem:

Theorem 1. *For $n = 2$ and $\alpha = 0$, BEST-EF-IS cannot be approximated up to any finite factor, unless P = NP.*

Theorem 2. *For $n = 2$ and $\alpha \in (0, 1)$, BEST-EF-IS is NP-hard.*

After establishing NP-hardness, our main results exhibit a sharp separation on the approximability between the cases of $n = 2$ and $n \geq 3$ agents. First, we can easily observe that the basic EF-IS allocation (i.e., where all items are sold) is an α-approximation of BEST-EF-IS, and we prove that, with at least three agents, no polynomial time algorithm can obtain a solution with a better approximation than that achieved by the basic EF-IS allocation, unless P = NP.

Theorem 3. *For any $n \geq 3$ and $\alpha \in (0, 1)$, BEST-EF-IS cannot be approximated with a ratio better than $\alpha + \epsilon$, for any constant $\epsilon > 0$, unless P = NP.*

On the other hand, for two agents, we are able to design a polynomial time approximation scheme (PTAS), under the assumption that the market value of each item is not smaller than half of the common agents' value.

Theorem 4. *There is a PTAS for BEST-EF-IS with two agents when* $\alpha \geq 1/2$.

The idea behind the PTAS is to enumerate all partial allocations of the most valuable items, whose number is a constant depending on the desired approximation guarantee. Each such partial allocation, which consists of the two bundles assigned to the agents together with the bundle of sold items, is then completed processing the remaining items by non-increasing value. At every step, the next item is allocated to the agent having the lower valued bundle, until we reach a situation where it is possible to equalize the two bundles by using the money raised from the already sold items and from selling a subset of the not-yet-processed ones. The main technical effort is needed to show that, if this condition occurs, then the final allocation can be made envy-free at the expense of a negligible loss of social welfare, while, if the condition never occurs, then it is not possible to obtain an envy-free solution from the starting partial allocation.

A further positive result we obtain is the design of a dynamic programming algorithm which runs in polynomial time when the number of distinct item values is constant; this assumption is in line with several other recent works on fair-division, e.g., [6, 7].

Theorem 5. *Let* T *be the number of distinct item values. BEST-EF-IS can be solved in time* $O(n(m/T)^{2T}T)$.

We then move to a generalization of our model, where agents can have heterogeneous valuations. Now, we assume that each agent i has her own additive valuation function v_i, so that $v_i(j)$ is the value of agent i for item j and $\boldsymbol{v}_i = (v_i(j))_{j\in[m]}$ denotes the vector of all item values for agent i. Under heterogeneous valuation functions, we need to be more careful about the market value vector $\boldsymbol{v}_0$. As also done in [5], we assume that for every item j, the market value satisfies $v_0(j) \leq min_i v_i(j)$. We find this a minimal assumption, that should hold so that no agent can have more value by selling an item rather than by owning it.

Let $\alpha := \min_{i\in[n],j\in[m]:v_i(j)>0} \left\{\frac{v_0(j)}{v_i(j)}\right\} \in [0,1]$ be the parameter defined similarly as in the case of identical valuations. As before, the basic EF-IS allocation is a feasible solution and trivially constitutes an optimal one when $\alpha = 1$. Moreover, the hardness results discussed above continue to hold under heterogeneous valuations. Therefore the problem is NP-hard, and with 3 agents or more, there is no approximation factor better than α. Nevertheless, we are still able to provide some positive results under certain assumptions. In particular, let $\beta := \max_{i\in[n]} \frac{\max_{j\in[m]} v_i(j)}{\min_{j\in[m]:v_i(j)>0} v_i(j)}$ denote the maximum ratio between the highest and the lowest (non-zero) valuable item of any agent. We obtain a PTAS, if n, β and $1/\alpha$ are bounded by a constant.

Theorem 6. *There is a PTAS for BEST-EF-IS with* n *heterogeneous agents, when* n, β *and* $1/\alpha$ *are all* $O(1)$.

This can be seen as generalizing the PTAS obtained for the case of identical valuations, even beyond the two agent case, but only with a constant β. The technique that we use however is quite different from that considered for two identical agents, and is is based on an appropriate combination of two main ideas. First, by using a linear programming formulation, we compute a fractional solution with a bounded number of fractionally assigned items. Then, we apply a "reverse" version of the envy cycle elimination algorithm [8], so as to decide which items to sell, in addition to the fractional ones. We believe that this could be of independent interest for other allocation problems as well.

Finally, we drop the assumption on β being constant and we also provide a pseudo-polynomial time algorithm.

Theorem 7. *BEST-EF-IS can be solved in $O(mn^2V^{n^2})$ time for heterogeneous valuations and in $O(mnV^n)$ time for identical ones, where $V = \max_{i\in[n]}\{v_i([m])\}$ denotes the maximum value for the entire set of items.*

4. Conclusions and Future Works

Our work explores from an algorithmic perspective the model of fair division of indivisible items initiated in [5], and provides an almost complete picture on its status. This model considers the possibility of selling items in order to compensate envious agents in a proposed allocation.

Despite the large amount of research work devoted in the last years to the study of relaxed notions of envy-freeness, the approach of items sale has remained largely unexplored. This may look strange since, although relaxed notions of envy-freeness such as EFX and EF1 provide theoretically interesting and elegant solutions to the non-existence of envy-free allocations, from a practical point of view there are many cases in which these solutions are highly unfair (think, for instance, of the famous basic case of a high-valued item and two agents). A possible reason for this under-consideration might come from the intrinsic difficulty of the problem, as witnessed by the strong computational barriers we proved in this work. However, we have also shown that, under some (in some cases even mild) assumptions, interesting positive results are possible.

An interesting open question that arises is whether we can extend the existence of a PTAS for two agents, in the case of $\alpha \in (0, 1/2)$ and identical valuations, and also in the case of arbitrary α and heterogeneous valuations (without further assumptions on other parameters). Furthermore, it would be nice to study the effects of items sale for other variants of fair allocation problems, such as for other notions of fairness (e.g., proportionality, EFX or maximin shares) or for more general valuations beyond additivity, or for problems with additional constraints (e.g., under connectivity constraints [9, 10, 11]). Finally, it would be interesting to study the case of strategic agents, as in [12], who may misreport their valuations to increase their utility.

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